

# On the total $(k, r)$ -domination number of random graphs

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A subset  $S$  of a vertex set of a graph  $G$  is a *total  $(k, r)$ -dominating set* if every vertex  $u \in V(G)$  is within distance  $k$  of at least  $r$  vertices in  $S$ . The minimum cardinality among all total  $(k, r)$ -dominating sets of  $G$  is called the *total  $(k, r)$ -domination number* of  $G$ , denoted by  $\gamma_{(k,r)}^t(G)$ . We previously gave an upper bound on  $\gamma_{(2,r)}^t(G(n, p))$  in random graphs with non-fixed  $p \in (0, 1)$ . In this paper we generalize this result to give an upper bound on  $\gamma_{(k,r)}^t(G(n, p))$  in random graphs with non-fixed  $p \in (0, 1)$  for  $k \geq 3$  as well as present an upper bound on  $\gamma_{(k,r)}^t(G)$  in graphs with large girth.

**Keywords:** random graphs, total  $(k, r)$ -domination

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## 1 Introduction

In this paper we derive upper bounds on the total  $(k, r)$ -domination number in graphs with large girth as well as in *random graphs*. A *random graph*  $G(n, p)$  consists of  $n$  vertices with each of the potential  $\binom{n}{2}$  edges being inserted independently with probability  $p$ . Random graphs can be used to model wireless sensor networks (WSNs), where sensors cooperatively collect data to monitor physical or environmental conditions. Generally WSNs are constructed in unreachable terrain and sensors may be arranged stochastically, which introduces uncertainty and randomness in the network structure. Distance and multiple domination have been used in the literature to address problems in wireless networks, such as area monitoring, fault tolerance in wireless sensor networks (WSNs). Thus, for positive integers  $k$  and  $r$ , a total  $(k, r)$ -dominating set in random graphs is a natural candidate to address area monitoring and fault tolerance in WSNs, where robustness for dominators is achieved by choosing a value for  $r > 1$  and the distance parameter  $k$  allows increasing local availability by reducing the distance to the dominators [13, 22, 26, 33].

The rest of the paper is organized as follows. In Section 2 we present a literature survey regarding upper bounds on the domination number and its variants. Section 3 derives an upper bound on  $\gamma_{(k,r)}^t(G)$  in graphs of large girth, and Section 4 derives an upper bound on  $\gamma_{(k,r)}^t(G(n, p))$  for  $k \geq 3$  and non-fixed  $p \in (0, 1)$  in random graphs.

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\*Research supported by FQRNT (Le Fonds de Recherche du Québec - Nature et Technologies) Doctoral Scholarship.

## 2 Related Work

Distance and multiple domination have been studied extensively by several authors [7, 8, 11, 17, 21, 23, 24, 27, 28, 29]. In [16, 25] upper bounds are given on the  $r$ -tuple domination number. A set  $D \subseteq V(G)$  is a  $r$ -tuple dominating set of  $G$  if for every vertex  $v \in V(G)$ ,  $|N[v] \cap D| \geq r$ , where  $N[v] = \{u \in V(G) | (u, v) \in E(G)\} \cup \{v\}$ . The minimum cardinality of a  $r$ -tuple dominating set of  $G$  is the  $r$ -tuple domination number of  $G$ , denoted  $\gamma_{\times r}(G)$ . Chang [9] further improved these results for any positive integer  $r$  and for any graph of  $n$  vertices with minimum degree  $\delta$ , where  $\gamma_{\times r}(G) \leq \frac{\ln(\delta - r + 2) + \ln \tilde{d}_{r-1} + 1}{\delta - r + 2}n$ , and  $\tilde{d}_m = \frac{1}{n} \sum_{i=1}^n \binom{d_i + 1}{m}$  with  $d_i$  being the degree of the  $i$ th vertex of  $G$ .

In [8] Caro and Yuster give upper bounds on the  $r$ -tuple and total  $r$ -domination numbers. A set  $D \subseteq V(G)$  is a total  $r$ -dominating set of  $G$  if for every vertex  $v \in V(G)$ ,  $|N(v) \cap D| \geq r$ , where  $N(v) = \{u \in V(G) | (u, v) \in E(G)\}$ . The minimum cardinality of a total  $r$ -dominating set of  $G$  is the total  $r$ -domination number of  $G$ , denoted  $\gamma_{\times r}^t(G)$ . In [34] Zhao et al. study the total  $r$ -domination number in graphs.

**Theorem 1.** [34] *In a graph  $G$  of order  $n$  and minimum degree  $\delta \geq r$ , where  $r \in \mathbb{N}$ , if  $\frac{\delta}{\ln \delta} \geq 2r$ , then  $\gamma_{\times r}^t(G) \leq \frac{n}{\delta} \left( r \ln \delta + \sum_{i=0}^{r-1} \frac{r-i}{i! \delta^{r-1-i}} \right)$ .*

Some works in the literature study upper bounds on the  $(k, r)$ -domination number. A set  $D \subseteq V(G)$  is a  $(k, r)$ -dominating set of  $G$  if for every vertex  $v \in V(G) \setminus D$  is within distance  $k$  of  $r$  vertices in  $D$ . The minimum cardinality of a  $(k, r)$ -dominating set of  $G$  is the  $(k, r)$ -domination number of  $G$ , denoted  $\gamma_{k,r}(G)$ . In [2] Bean et al. posed the following conjecture.

**Conjecture 1.** [2] *Let  $G$  be a graph of order  $n$  and let  $\delta_k$  denote the smallest cardinality among all  $k$ -neighbourhoods of  $G$ , where  $\delta_k \geq k + r - 1$ . Then for positive integers  $k$  and  $r$   $\gamma_{(k,r)}(G) \leq \frac{r}{r+k}n$ .*

Fischermann and Volkmann confirmed that the conjecture is valid for all integers  $k$  and  $r$ , where  $r$  is a multiple of  $k$  [15]. In [20] Korneffel et al. show that  $\gamma_{2,2}(G) \leq \frac{n(G)+1}{2}$ .

There are several works in the literature that study upper bounds on the domination number and its variants in random graphs. Recall that a random graph  $G(n, p)$  consists of  $n$  vertices with each of the potential  $\binom{n}{2}$  edges being inserted independently with probability  $p$ . We say that an event holds *asymptotically almost surely (a.a.s)* if the probability that it holds tends to 1 as  $n$  tends to infinity.

Dreyer [14] in his dissertation studied the question of domination in random graphs. Wieland and Godbole proved that the domination number of a random graph, denoted  $\gamma(G(n, p))$ , has a two point concentration [32].

**Theorem 2.** [32] *For  $p \in (0, 1)$  fixed, a.a.s  $\gamma(G(n, p))$  equals  $\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1$  or  $\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$ , where  $\mathbb{L}n = \log_{1/(1-p)} n$ .*

Wang and Xiang [30] extend this result for 2-tuple domination number of  $G(n, p)$ .

**Theorem 3.** [30] *For  $p \in (0, 1)$  fixed, a.a.s  $\gamma_{\times 2}(G(n, p))$  equals  $\left\lfloor \mathbb{L}n - \mathbb{L}(\log n) + \mathbb{L}\left(\frac{p}{1-p}\right) \right\rfloor + 1$  or  $\left\lfloor \mathbb{L}n - \mathbb{L}(\log n) + \mathbb{L}\left(\frac{p}{1-p}\right) \right\rfloor + 2$ , where  $\mathbb{L}n = \log_{1/(1-p)} n$ .*

Bonato and Wang [6] study the total domination number and the independent domination number in random graphs. For a graph  $G$ , a set  $D \subseteq V(G)$  is an independent dominating set of  $G$  if  $D$  is both an

independent set and a dominating set of  $G$ . The *independent domination number* of  $G$ , denoted  $\gamma_i(G)$ , is the minimum order of an independent dominating set of  $G$ .

**Theorem 4.** [6] For  $p \in (0, 1)$  fixed, a.a.s  $\gamma_t(G(n, p))$  equals  $\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1$  or  $\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$ , where  $\mathbb{L}n = \log_{1/(1-p)} n$ .

**Theorem 5.** [6] For  $p \in (0, 1)$  fixed, a.a.s  $\lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1 \leq \gamma_i(G(n, p)) \leq \lfloor \mathbb{L}n \rfloor$ , where  $\mathbb{L}n = \log_{1/(1-p)} n$ .

Wang further studied the independent domination number of random graphs [31].

**Theorem 6.** [31] Let  $p \in (0, 1)$  and  $\epsilon \in (0, \frac{1}{2})$  be two real numbers. Let  $k = k(p, \epsilon) \geq 1$  be the smallest integer satisfying  $(1-p)^k < \frac{1}{2} - \epsilon$ . A.a.s.  $\gamma(G(n, p)) \leq \gamma_i(G(n, p)) \leq \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + k + 1$ , where  $\mathbb{L}n = \log_{1/(1-p)} n$ .

If  $p > \frac{1}{2}$ , then for  $\epsilon \in (0, p - \frac{1}{2}) \subset (0, \frac{1}{2})$ , by Theorems 2 and 6, the following concentration result follows.

**Corollary 1.** [31] For  $p \in (\frac{1}{2}, 1)$  fixed, a.a.s.  $\gamma(G(n, p)) \leq \gamma_i(G(n, p)) \leq \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2$ , where  $\mathbb{L}n = \log_{1/(1-p)} n$ .

[18] studies an upper bound on  $\gamma_{(2,r)}^t(G(n, p))$  in random graphs for non-fixed  $p \in (0, 1)$ .

**Theorem 7.** [18] Let  $c > 1$  be a fixed constant. Then for any positive integer  $r$ , in a random graph  $G(n, p)$  with  $p \geq c\sqrt{\frac{\log n}{n}}$ , a.a.s.  $\gamma_{(2,r)}^t(G(n, p)) = r + 1$ .

To the best of our knowledge, there are no works in the literature that study the upper bounds on the total  $(k, r)$ -domination number in general graphs or in random graphs. In this paper we give an upper bound on  $\gamma_{(k,r)}^t(G)$  in graphs of large girth and extend the results of [18] to derive an upper bound on  $\gamma_{(k,r)}^t(G(n, p))$  in random graphs.

### 3 Total $(k, r)$ -domination number in graphs of large girth

In this section we derive an upper bound on the total  $(k, r)$ -domination number in graphs with large girth. We present our result in Theorem 8. Although our result is not tight, we do obtain a bound with relatively simple expression.

**Theorem 8.** Consider a graph  $G$ , where  $n = |V(G)|$ . Let  $G$  be of minimum degree at least  $d$ , and of girth at least  $2k + 1$ . Then for any positive integers  $k$  and  $r$ ,  $\gamma_{(k,r)}^t(G) \leq \frac{2nr}{(d-1)^k} + nre^{-\frac{r}{4}}$ .

**Proof:**

Let us pick, randomly and independently, each vertex  $v \in V(G)$  with probability  $p$ . Let  $S \subset V(G)$  be the set of vertices picked. We will determine the value of  $p$  by the end of the proof.  $S$  is a random set and is part of the total  $(k, r)$ -dominating set that we would like to obtain.

Let the distance from a vertex  $u$  to a vertex  $v$  be denoted as  $d(u, v)$ , which is the length of the shortest path between  $u$  and  $v$ . For every vertex  $v \in V(G)$ , let  $X_v$  denote the number of vertices in  $N_k(v)$  that are also in  $S$ , where  $N_k(v) = \{u \in V \mid u \neq v \text{ and } d(u, v) \leq k\}$ . Let  $Y$  be the set such that

$Y = \{v \in V(G) | X_v \leq r - 1\}$ . Note that  $S$  is a random set and  $\mathbb{E}[|S|] = np$ . We now estimate  $\mathbb{P}[X_v < r]$ .

For a given vertex  $v \in Y$ , let  $m = |N_k(v)|$ . We will show by contradiction that  $m \geq (d - 1)^k$ .

Assume that  $m < (d - 1)^k$ . Then there exist vertices  $u_1, u_2 \in N_k(v)$  such that there is a vertex  $w \in N_k(v)$  and  $w \in (N_k(u_1) \cap N_k(u_2))$ . Vertex  $w$  is at most distance  $k$  from  $v$ . Thus, the distance from  $w$  to  $v$  through the path containing  $u_1$  is at most  $k$ . Similarly, the distance from  $w$  to  $v$  through the path containing  $u_2$  is also at most  $k$ . Thus, making a cycle of length at most  $2k$ , which is a contradiction. Therefore, by the assumption that  $G$  has girth at least  $2k + 1$ , it follows that  $m \geq (d - 1)^k$ .

It can be seen that  $X_v$  is a  $B(m, p)$  random variable. We use the well known Chernoff Bound [4, 12, 10, 3, 19, 1] to bound  $\mathbb{P}[X_v \leq r - 1] = \mathbb{P}[X_v < r]$ .

The Chernoff Bound states: for any  $a > 0$  and random variable  $X$  that has binomial distribution with probability  $p$  and mean  $pn$ ,

$$\mathbb{P}[X - pn < -a] < e^{-a^2/2pn}. \quad (1)$$

We set  $a = \epsilon pm$ , where we let  $\epsilon = 1 - \frac{r}{pm}$ . Hence,  $a = pm - r$ , which results in  $r = pm - a$ . Then, by the Chernoff Bound given in Equation 1 we have,

$$\begin{aligned} \mathbb{P}[X_v < r] &= \mathbb{P}[X_v < pm - a] \\ &< e^{-\frac{a^2}{2pm}} = e^{-\frac{\epsilon^2(pm)^2}{2pm}} = e^{-\frac{\epsilon^2 pm}{2}} \\ &\leq e^{-\frac{\epsilon^2 p(d-1)^k}{2}}. \end{aligned} \quad (2)$$

Chernoff's bound holds whenever  $\epsilon > 0$ , that is when  $1 - \frac{r}{pm} > 0$ . Thus, it holds when  $p > \frac{r}{(d - 1)^k}$ .

By setting  $p = \frac{2r}{(d - 1)^k}$ , from Equation 2 we obtain

$$\mathbb{P}[X_v < r] < e^{-\epsilon^2 \left(\frac{2r}{(d-1)^k}\right) \frac{(d-1)^k}{2}} = e^{-\epsilon^2 r}.$$

For each vertex  $v \in Y$ , where  $X_v \leq r - 1$ , we pick a set  $A_v$  of  $r$  vertices in  $N_k(v)$  arbitrarily. For vertices  $v$  that satisfy  $X_v \geq r$ ,  $A_v = \emptyset$ . Let  $A = \bigcup_{v=1}^n A_v$ . Clearly,  $S \cup A$  is a total  $(k, r)$ -dominating set.

We now estimate  $\mathbb{E}[|A|]$ . By linearity of expectation, we obtain

$$\begin{aligned} \mathbb{E}[|A|] &= \mathbb{E}\left[\left|\bigcup_{v=1}^n A_v\right|\right] \leq \mathbb{E}\left[\sum_{v=1}^n |A_v|\right] \\ &= \sum_{v=1}^n \mathbb{E}[|A_v|] \leq nre^{-\epsilon^2 r}. \end{aligned}$$

Again by the linearity of expectation, we now estimate  $\mathbb{E}[|S \cup A|]$ .

$$\begin{aligned}\mathbb{E}[|S \cup A|] &= \mathbb{E}[|S|] + \mathbb{E}[|A|] \\ &\leq np + nre^{-\epsilon^2 r} \\ &= \frac{2nr}{(d-1)^k} + nre^{-\epsilon^2 r}.\end{aligned}$$

Therefore, we have shown that there exists a total  $(k, r)$ -dominating set in  $G$ , where

$$\begin{aligned}\gamma_{(k,r)}^t(G) &\leq \frac{2nr}{(d-1)^k} + nre^{-\epsilon^2 r} \\ &\leq \frac{2nr}{(d-1)^k} + nre^{-\frac{r}{4}}\end{aligned}$$

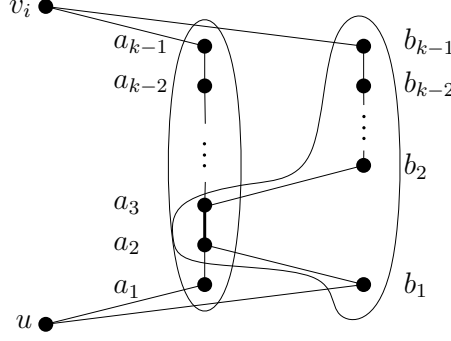
since  $\epsilon > 1/2$ . □

## 4 Total $(k, r)$ -domination number in random graphs

In [18], we presented an upper bound on the total  $(2, r)$ -domination number of the random graphs. In this section we generalize this result to derive an upper bound on the total  $(k, r)$ -domination number in random graphs for  $k \geq 3$ . However, before doing so we briefly discuss the main difference between the solutions of  $\gamma_{(2,r)}^t(G(n, p))$  and  $\gamma_{(k,r)}^t(G(n, p))$ , for  $k \geq 3$ .

In Theorem 7 it is proved that in a random graph  $G(n, p)$  with  $p \geq c\sqrt{\frac{\log n}{n}}$  and a fixed constant  $c > 1$ , a.a.s.  $\gamma_{(2,r)}^t(G(n, p)) = r + 1$ . In the proof of Theorem 7 it is needed to calculate the probability that a vertex  $u$  is not within distance-2 from a dominator vertex  $v_i$ , i.e.  $\mathbb{P}[v_i \notin N_2(u)]$ . To connect  $u$  to  $v_i$  via a path of length 2, one connecting vertex, denoted  $w_i$ , is needed. To determine that  $\mathbb{P}[v_i \notin N_2(u)]$  uses the fact that the edges between  $u, w_i$  and  $v, w_i$  that connect  $u$  to  $v_i$  (in order to obtain a path of length 2 and  $u$  to be dominated by  $v_i$ ) cannot be chosen again to connect  $u$  to  $v_i$  via a different path (since the two paths would be the same). Hence, the probability that there exists an edge between any two connecting vertices among all different paths of length 2 from  $u$  to  $v_i$  are independent of each other. So, the probability that there is a path from  $u$  to  $v_i$  via a given connecting vertex  $w_i$  is  $p^2$ . There are  $n - 2$  such vertices and thus, the probability that a vertex  $u$  is not within distance-2 from a dominator vertex  $v_i$  is given by  $\mathbb{P}[v_i \notin N_2(u)] \leq (1 - p^2)^{(n-2)}$ .

Similar calculation is needed in the proof of Theorem 11 that follows to determine the probability that a vertex  $u$  is not within distance- $k$  from a dominator vertex  $v_i$  (Theorem 11, Equation 9). However, once we generalize to give an upper bound on the total  $(k, r)$ -domination number, we cannot easily obtain an independence between the existing edges of any  $k - 1$  connecting vertices among all the different paths of length  $k$  from  $u$  to  $v_i$ . When considering paths of length  $k$  from  $u$  to  $v_i$  for the general case of total  $(k, r)$ -domination number it becomes more difficult to calculate the probability that there is a path of length  $k$  from  $u$  to  $v_i$  via  $k - 1$  vertices. There may be two different paths  $P_1$  and  $P_2$  from  $u$  to  $v_i$  that may share some edges between any of the connecting  $k - 1$  vertices and hence, are not independent anymore as they were in the case of total  $(2, r)$ -domination number (see Fig. 1).



**Fig. 1:**  $P_1 = u a_1 a_2 a_3 \cdots a_{k-2} a_{k-1} v_i$  and  $P_2 = u b_1 a_2 a_3 b_2 \cdots b_{k-2} b_{k-1} v_i$  are two paths between  $u$  and  $v_i$  that share an edge, namely  $(a_2, a_3)$ .

Bollobás has the following result on random graphs of diameter greater than two.

**Theorem 9.** [5] Let  $c$  be a positive constant,  $d = d(n) \geq 2$  a natural number, and define  $p = p(n, c, d)$ ,  $0 < p < 1$ , by

$$p^d n^{d-1} = \log(n^2/c).$$

Suppose that  $pn/(\log n)^3 \rightarrow \infty$ . Then in  $G(n, p)$  we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(\text{diam } G = d) = e^{-c/2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{P}(\text{diam } G = d + 1) = 1 - e^{-c/2}.$$

Note that from Theorem 9, the diameter of  $G(n, p)$  is at most  $d$  for  $p = \sqrt[d-1]{\frac{\log(n^2/c)}{n^{d-2}}}$ . In Theorem 11, we weaken the value of  $p$  to  $p' \geq d \sqrt[d]{\frac{\log n}{n^{d-1}}}$ . Easy calculation shows that  $d \sqrt[d]{\frac{\log n}{n^{d-1}}} < \sqrt[d-1]{\frac{\log(n^2/c)}{n^{d-2}}}$  for  $d \leq (\log n)^\epsilon$  for a constant  $\epsilon < \frac{1}{2}$ . In particular,  $p' < p$  holds when  $d$  is constant. Our proof of Theorem 11 uses Janson's Inequality, which we present here first [1].

Let  $\Omega$  be a finite universal set and let  $R$  be a random subset of  $\Omega$  given by

$$\mathbb{P}[r \in R] = p_r, \tag{3}$$

these events are mutually independent over  $r \in \Omega$ . Let  $\{A_i\}_{i \in I}$  be subsets of  $\Omega$ , where  $I$  is a finite index set. Let  $B_i$  be the event that  $A_i \subseteq R$ . Let  $X_i$  be the indicator random variable for  $B_i$  and  $X = \sum_{i \in I} X_i$

the number of  $A_i \subseteq R$ . Hence,  $\mathbb{P}[X = 0] = \mathbb{P}\left[\bigcap_{i \in I} \overline{B_i}\right]$ . For  $i, j \in I$  we write  $i \sim j$  if  $i \neq j$  and  $A_i \cap A_j \neq \emptyset$ . Thus, define  $\Delta = \sum_{i \sim j} \mathbb{P}[B_i \cap B_j]$ . Set  $\mu = \mathbb{E}[X] = \sum_{i \in I} \mathbb{P}[B_i]$ . In Theorem 10 we state Janson's Inequality. [1]

**Theorem 10.** [1] Let  $\{B_i\}_{i \in I}$ ,  $\Delta$ ,  $\mu$  be as above. Then  $\mathbb{P}\left[\bigcap_{i \in I} \overline{B_i}\right] \leq e^{-\mu + \Delta/2}$ .

Now, we present our main result on the total  $(k, r)$ -domination number of the random graphs.

**Theorem 11.** For any positive integers  $k \geq 3$  and  $r$ , in a random graph  $G(n, p)$  with  $p \geq k \sqrt{\frac{\log n}{n^{k-1}}}$ , a.a.s.  $\gamma_{(k,r)}^t(G(n, p)) = r + 1$ .

**Proof:** Let  $D \subseteq V(G(n, p))$  be a total  $(k, r)$ -dominating set and let the vertices in  $D$  be labelled as  $v_1, v_2, \dots, v_i, \dots, v_{r+1}$ , where  $1 \leq i \leq r + 1$ . The probability that a vertex  $u \in V(G(n, p))$  is not within distance- $k$  from a vertex  $v_i \in D$  is denoted by  $\mathbb{P}[v_i \notin N_k(u)]$ .

Let  $X$  be a random variable that denotes the number of vertices  $u \in V(G(n, p))$ , where the number of  $k$ -adjacent vertices of  $u$  in  $D$  is less than  $r$ . We would like to show that the number of vertices in  $V(G(n, p))$  with less than  $r$  dominators tends to 0. That is,  $\mathbb{P}[X > 0] \rightarrow 0$  as  $n \rightarrow \infty$ .

We define a fixed vertex  $u$  as *bad*, if  $u$  in its  $k$ -neighborhood has less than  $r$  dominators in  $D$ . By linearity of expectation we have

$$\mathbb{E}[X] = n \cdot \mathbb{P}[\text{fixed } u \text{ is bad}]. \quad (4)$$

There are  $n - 2$  vertices aside from  $u$  and  $v_i$  to connect  $u$  to  $v_i$  via a path of length  $k$ . To connect  $u$  to  $v_i$  such that  $d(u, v_i) = k$ , additional  $k - 1$  connecting vertices are necessary to create a path of length  $k$  from  $u$  to  $v_i$ . There are  $\binom{n-2}{k-1}$  possible ways to choose these  $k - 1$  vertices. Hence, we have  $\binom{n-2}{k-1}$  such sets that consist of  $k - 1$  vertices. We denote these sets by  $S_1, S_2, \dots, S_{\binom{n-2}{k-1}}$ .

We would like to show that  $\mathbb{P}[v_i \notin N_k(u)] \rightarrow 0$  as  $n \rightarrow \infty$ . This is equivalent to showing that the probability one of  $S_i$  connects  $u$  to  $v_i$  via a path of length  $k$  tends to 1 as  $n \rightarrow \infty$ .

Let  $S_i = \{a_{i_1}, a_{i_2}, \dots, a_{i_{k-1}}\}$ . For any pair  $u$  and  $v_i$  that are fixed, we number all other  $n - 2$  vertices and assume that all vertices in  $S_i$  are connected in ascending order of the vertex number. Note that some edges in  $S_i$  and  $S_j$ , where  $i \neq j$  are the same. To calculate the probability of the appearance of the  $k - 2$  edges in each  $S_i$  we must consider the dependencies between any two sets  $S_i$  and  $S_j$  for  $i \neq j$ . To do this, we use Janson's inequality from Theorem 10.

Let  $R$  be the set  $E(G(n, p))$  and let  $A_i$  be the set of edges such that  $A_i = \{ua_{i_1}, a_{i_1}a_{i_2}, \dots, a_{i_{k-2}}a_{i_{k-1}}, a_{i_{k-1}}v_i\}$ . Let  $B_i$  be the event that  $A_i \subseteq R$ . So,  $\mathbb{P}[A_i \in R] = \mathbb{P}[B_i]$ . Let  $X_i$  be the indicator random

variable for  $B_i$  and  $X_B = \sum_{i=1}^{\binom{n-2}{k-1}} X_i$  be the number of  $A_i \subseteq R$ . Hence,  $\mathbb{P}[X_B = 0] = \mathbb{P}\left[\bigcap_{i=1}^{\binom{n-2}{k-1}} \overline{B_i}\right]$ . For

$1 \leq i, j \leq \binom{n-2}{k-1}$  we write  $i \sim j$  if  $i \neq j$  and  $A_i \cap A_j \neq \emptyset$ .  $\Delta$  is defined as  $\sum_{i \sim j} \mathbb{P}[B_i \cap B_j]$ . We would

like to show that  $\mathbb{P}[X_B = 0] \rightarrow 0$  as  $n \rightarrow \infty$ .

First we determine  $\mu = \mathbb{E}[X_B] = \sum_{i=1}^{\binom{n-2}{k-1}} \mathbb{P}[B_i]$ .

$$\begin{aligned}
\mathbb{E}[X_B] &= \mathbb{E}\left[\sum_{i=1}^{\binom{n-2}{k-1}} X_i\right] = \sum_{i=1}^{\binom{n-2}{k-1}} \mathbb{E}[X_i] = \sum_{i=1}^{\binom{n-2}{k-1}} \mathbb{E}[B_i] \\
&= \binom{n-2}{k-1} p^k \geq \left(\frac{n-2}{k-1}\right)^{k-1} p^k \quad \left(\text{by } \binom{n}{k} \geq \left(\frac{n}{k}\right)^k\right) \\
&\geq \frac{(n-2)^{k-1}}{(k-1)^{k-1}} \left(k \sqrt[k]{\frac{\log n}{n^{k-1}}}\right)^k \\
&= \frac{(n-2)^{k-1}}{(k-1)^{k-1}} k^k \frac{\log n}{n^{k-1}} = \left(\frac{k^k}{(k-1)^{k-1}}\right) \left(\frac{n-2}{n}\right)^{k-1} \log n \\
&= k \left(\frac{k}{k-1}\right)^{k-1} \left(1 - \frac{2}{n}\right)^{k-1} \log n \\
&\geq k \left(1 - \frac{2}{n}\right)^{k-1} \log n \\
&\geq 0.9k \log n
\end{aligned} \tag{5}$$

for  $n$  large enough. Thus, in Janson's Inequality let  $\mu = 0.9k \log n$ .

Now we determine  $\Delta$ . Assume that the number of edges shared between any given  $A_i$  and  $A_j$  is given by  $t$  and hence,  $A_j$  shares at least  $t$  vertices with  $A_i$ . There are  $\binom{n-2}{k-1}$  such  $A_i$  sets. We fix one such set  $A_i$  and determine the dependencies between  $A_i$  and all other sets  $A_j$ , where  $j \neq i$ . Thus, we have

$$\begin{aligned}
\Delta &= \sum_{i=1}^{\binom{n-2}{k-1}} \mathbb{P}[B_i \cap B_j] \\
&\leq \binom{n-2}{k-1} \sum_{\substack{i \text{ fixed} \\ j \sim i}}^{\binom{n-2}{k-1}} \mathbb{P}[B_j \cap B_i] \\
&\leq \binom{n-2}{k-1} \sum_{t=1}^{k-1} \binom{k}{t} \binom{n-2}{k-1-t} p^{2k-t}.
\end{aligned} \tag{6}$$

In Equation 6, the probability that a fixed  $A_i$  intersects (i.e. shares) at  $t$  edges with a set  $A_j$  for  $i \neq j$ , is  $p^k p^{k-t} = p^{2k-t}$ . When calculating this probability we are interested in counting the number of edges  $t$  that are shared between  $A_i$  and  $A_j$ . That is, between which vertices  $t$  edges are shared is not of interest. Between any two vertices  $u$  and  $v_i$  there are  $k$  edges and hence, the number of ways to determine the  $t$  shared edges is  $\binom{k}{t}$ . For any  $A_j$ , the two vertices  $u$  and  $v_i$  are fixed. From the  $k-1$  other vertices on the path from  $u$  to  $v_i$ ,  $t$  are shared with  $A_i$ . Hence, to complete  $A_j$  that share  $t$  edges with  $A_i$ , there are  $\binom{n-2}{k-1-t}$  possible ways to add the remaining vertices. Thus, for a given value  $t$ ,  $\binom{k}{t} \binom{n-2}{k-1-t}$  determine



how many sets  $A_j$  share precisely  $t$  edges with  $A_i$ . Thus, from Equation 6 we have

$$\begin{aligned}
\Delta &\leq \binom{n-2}{k-1} \sum_{t=1}^{k-1} \binom{k}{t} \binom{n-2}{k-1-t} p^{2k-t} \\
&\leq \frac{n^{k-1}}{(k-1)!} 2^k \sum_{t=1}^{k-1} \binom{n-2}{k-1-t} p^{2k-t} \quad \left( \text{by } \binom{n}{k} \leq \frac{n^k}{k!} \text{ and } \binom{n}{k} \leq 2^n \right) \\
&\leq \frac{n^{k-1}}{(k-1)!} 2^k \sum_{t=1}^{k-1} \frac{(n-2)^{k-1-t}}{(k-1-t)!} p^{2k-t} \quad \left( \text{by } \binom{n}{k} \leq \frac{n^k}{k!} \right) \\
&\leq \frac{n^{k-1}}{(k-1)!} 2^k \sum_{t=1}^{k-1} n^{k-1-t} p^{2k-t}.
\end{aligned} \tag{7}$$

We now calculate  $n^{k-1-t} p^{2k-t}$ ,

$$\begin{aligned}
n^{k-1-t} p^{2k-t} &= \frac{n^{k-t}}{n} p^k p^{k-t} = \frac{n^{k-t} p^{k-t}}{n} p^k \\
&= \frac{n^{k-t} p^{k-t}}{n} \left( k \sqrt[k]{\frac{\log n}{n^{k-1}}} \right)^k = \frac{n^{k-t} p^{k-t}}{n} k^k \frac{\log n}{n^{k-1}} \\
&= \frac{n^{k-t} p^{k-t}}{n^k} k^k \log n = n^{-t} p^{k-t} k^k \log n \\
&= (np)^{-t} (p^k k^k \log n) = \left( nk \sqrt[k]{\frac{\log n}{n^{k-1}}} \right)^{-t} (p^k k^k \log n) \\
&= \left( n^{1-(k-1)/k} k \sqrt[k]{\log n} \right)^{-t} (p^k k^k \log n) \\
&= \left( n^{1/k} k \sqrt[k]{\log n} \right)^{-t} (p^k k^k \log n) = \frac{(p^k k^k \log n)}{(kn^{1/k} \sqrt[k]{\log n})^t} \\
&\leq \frac{(p^k k^k \log n)}{kn^{1/k} \sqrt[k]{\log n}}.
\end{aligned} \tag{8}$$

From Equations 7 and 8 we have

$$\begin{aligned}
\Delta &\leq \frac{n^{k-1}}{(k-1)!} 2^k \sum_{t=1}^{k-1} n^{k-1-t} p^{2k-t} \leq \frac{n^{k-1}}{(k-1)!} 2^k \sum_{t=1}^{k-1} \frac{p^k k^k \log n}{kn^{1/k} \sqrt[k]{\log n}} \\
&\leq \frac{n^{k-1}}{(k-1)!} 2^k k \frac{p^k k^k \log n}{kn^{1/k} \sqrt[k]{\log n}} \leq 2^k \frac{n^{k-1}}{(k-1)!} \left( k \sqrt[k]{\frac{\log n}{n^{k-1}}} \right)^k \frac{k^k \log n}{n^{1/k} \sqrt[k]{\log n}} \\
&\leq \frac{2^k}{(k-1)!} k^{2k} \frac{n^{k-1} \log^2 n}{n^{k-1} n^{1/k} \sqrt[k]{\log n}} \leq O(k) \frac{\log^2 n}{n^{1/k} \sqrt[k]{\log n}} \leq O(k) \frac{\log^2 n}{n^{1/k}}.
\end{aligned}$$

Thus,  $\Delta \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\Delta < \mu$  by Janson's Inequality we have

$$\mathbb{P}[X_B = 0] = \mathbb{P}\left[\bigcap_{i=1}^{\binom{n-2}{k-1}} \overline{B_i}\right] \leq e^{-\mu/2} \leq e^{-\frac{0.9k \log n}{2}} \leq e^{-\frac{9}{20}k \log n}.$$

Thus, the probability that a vertex  $u$  is not within distance- $k$  from a dominator vertex  $v_i$  is given by

$$\mathbb{P}[v_i \notin N_k(u)] \leq \mathbb{P}\left[\bigcap_{i=1}^{\binom{n-2}{k-1}} \overline{B_i}\right] \leq e^{-\frac{9}{20}k \log n}. \quad (9)$$

Let  $X_u$  be the random variable that denotes the number of non-dominators of  $u$ . We note that  $u$  may be a dominating vertex. Then

$$\mathbb{E}[X_u] \leq r e^{-\frac{9}{20}k \log n}.$$

By Markov's Inequality we have  $\mathbb{P}[X_u > 0] \leq \mathbb{E}[X_u] \leq r e^{-\frac{9}{20}k \log n}$ . Thus,

$$\mathbb{P}[\text{fixed } u \text{ is bad}] \leq \mathbb{P}[X_u > 0] \leq r e^{-\frac{9}{20}k \log n}. \quad (10)$$

By Equation 4 and Equation 10 we have  $\mathbb{E}[X] \leq n r e^{-\frac{9}{20}k \log n}$  and by Markov's Inequality it follows,

$$\mathbb{P}[X > 0] \leq \mathbb{E}[X] \leq n r e^{-\frac{9}{20}k \log n}. \quad (11)$$

From Equation 11, we determine the value of  $e^{-\frac{9}{20}k \log n}$  to be

$$e^{-\frac{9}{20}k \log n} \geq (e^{\log n})^{-\frac{9}{20}k} = n^{-\frac{9}{20}k}.$$

Thus, we have

$$n r e^{-\frac{9}{20}k \log n} \leq \frac{nr}{n^{\frac{9}{20}k}} \leq \frac{r}{n^{\frac{9}{20}k-1}}.$$

For  $k \geq 3$ ,  $\frac{r}{n^{\frac{9}{20}k-1}} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $\mathbb{P}[X > 0] \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

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